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3-Manifolds with irreducible Heegaard splittings of high genus

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Abstract

Non-isotopic Heegaard splittings of non-minimal genus were known previously only for very special 3-manifolds. We show in this paper that they are in fact a widespread phenomenon in 3-manifold theory: We exhibit a large class of knots and manifolds obtained by Dehn surgery on these knots which admit such splittings. Many of the manifolds have irreducible Heegaard splittings of arbitrary large genus. All these splittings are horizontal and are isotopic, after one stabilization, to a multiple stabilization of certain canonical low genus vertical Heegaard splittings. © 2000 Published by Elsevier Science Ltd. All rights reserved.

0. Introduction

Every closed orientable 3-manifold M has a *Heegaard splitting* which is a decomposition of M along an orientable surface $\Sigma \subset M$ into two handlebodies H_1, H_2 . The genus of this *Heegaard surface* Σ is called the *genus* of the splitting. There is a canonical process, called *stabilization*, which transforms a Heegaard splitting of genus g into one of genus $g + 1$. If M is irreducible, then a Heegaard splitting $M = H_1 \cup_{\Sigma} H_2$ is *irreducible* if it is not obtained from another splitting of lower genus by stabilization. A detailed review of these notions and facts is given below in Section 1.

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The set $\mathcal{H}(M)$ of all isotopy classes of Heegaard splittings for a given 3-manifold M could be determined so far only for a small number of “simple” manifolds (see the discussion in Section 1). Still, it is known for many manifolds M that there is more than one isotopy class of *minimal genus* Heegaard splittings (see [9,10]). For Seifert fibered spaces all irreducible Heegaard splittings are classified into two types: They are either *vertical* or *horizontal* (see Definition 1.2). There is accumulating evidence that a similar classification might be true for hyperbolic manifolds (see [4,13,14] and the discussion in Section 1.)

For *non-minimal* genus Heegaard splittings very little is known. The only manifolds M for which non-isotopic Heegaard splittings of non-minimal genus have been exhibited are obtained by surgery on pretzel knots [4], or by Casson–Gordon Kobayashi for torus sum of pretzel link complements with 2-bridge link complements [8]. In both cases M is shown to contain irreducible Heegaard splittings of arbitrarily large genus.

In this paper we define *vertical* and *horizontal* Heegaard splittings in a broad context, which generalizes the above-mentioned earlier notions. Our results stated below show that the results of [4,8] about high genus irreducible Heegaard splittings are only the first examples of a phenomenon which is in fact widespread among 3-manifolds, and which is based on the existence of high genus horizontal Heegaard surfaces of pairs.

For a general 3-manifold M and a link $K \subset M$ we introduce the notion of a Heegaard splitting of a pair (M, K) which can be vertical or horizontal (see Section 1.1). A vertical Heegaard splitting of (M, K) will always induce a Heegaard splitting on all manifolds obtained by surgery on K . However, a horizontal Heegaard splitting Σ of (M, K) , will induce a Heegaard splitting only on manifolds obtained by specific surgeries. Nevertheless we show that horizontal Heegaard splittings are quite common.

Recall that every knot or link $K \subset S^3$ is isotopic to a $2n$ -plat (see Fig. 8) of length m , for some $m, n \in \mathbb{N}$, and that every such $2n$ -plat can be described by a family of parameters $a_{i,j} \in \mathbb{Z}$, called *twist coefficients* (see Section 4). Summing up a well-defined subset of these twist coefficients (see Section 4) we compute a *plat linking number* $a(K) \in \mathbb{Z}$. To every closed 3-manifold obtained by surgery on a knot K , given as a $2n$ -plat, there are two canonically associated Heegaard surfaces Σ_{top} and Σ_{bot} of genus n (see Section 5). Let $M = K(p/q)$ denote the closed 3-manifold obtained from $\frac{p}{q}$ -surgery on K .

Theorem 0.1. *Let K be a knot given as a $2n$ -plat in S^3 , and assume that all twist coefficients satisfy $|a_{i,j}| \geq 3$. Then for all $k \in \mathbb{Z}$, with $|k| \geq 6$, the manifold $K((1 + ka(K))/k)$ has an irreducible Heegaard splitting of genus $m(n - 1)$. Furthermore, all these Heegaard splittings are horizontal.*

The main tool in this paper is a new combinatorial object called a *trellis* (see Section 2) which generalizes the notion of a $2n$ -plat and allows us to present a knot or link by a family of integer parameters, assembled in a *twist matrix*. Again, we can compute an analogous *trellis linking number* $a(K)$. For every knot K , carried by a trellis T , we obtain a *trellis Heegaard splitting* of genus $g(T)$ for the pair (S^3, K) and for the surgery manifold $K((1 + ka(K))/k)$. If we consider trellisses with a particular combinatorial feature, called an *interior pair of edges*, we can perform flypes on these more general knots in a way similar to that done by Casson–Gordon in [4] for pretzel knots (see Section 3). This allows us to show an analogous result for a rather large class of 3-manifolds:

Theorem 0.2. *Let T be a generalized trellis and let $K = K(A) \subset S^3$ be a knot carried by T with twist matrix A . Assume that all coefficients $a_{i,j}$ of A satisfy $|a_{i,j}| \geq 3$ and that there is an interior pair of edges $(e_{i,j}, e_{i,h})$ of T with twist coefficients $|a_{i,j}|, |a_{i,h}| \geq 4$. Then for all $k, n \in \mathbb{Z}$, with $|k| \geq 6$, the manifolds $K((1 + ka(K))/k)$ have irreducible Heegaard splittings $\Sigma(n)$ of arbitrarily large genus $g(T) + 2n$, all of which are horizontal.*

The above theorems seem, at first sight, to squelch the hope for a natural structure theorem concerning the set $\mathcal{H}(M)$ of all isotopy classes of Heegaard splittings for M . However, the following result perhaps resurrects some of these hopes:

Theorem 0.3. *Let T be a generalized trellis and $K \subset S^3$ a knot carried by T . Then for all $k \in \mathbb{Z}$ the trellis Heegaard splitting of $K(1 + ka(K)/k)$ is isotopic, after one stabilization, to a multiple stabilization of the canonical top Heegaard splitting Σ_{top} (and also of Σ_{bot}) defined by K . In particular, for K as in Theorem 0.2, all of the splittings $\Sigma(n)$, stabilized once, are stabilizations of a common low genus Heegaard splitting.*

Here Σ_{top} and Σ_{bot} are low genus *vertical* Heegaard splittings of $M = K((1 + ka(K))/k)$ with respect to the core curve K' of the surgery filling torus. They generalize the canonical top and bottom splittings for $2n$ -plats (see Section 5). It has been shown in [10] that for sufficiently complicated $2n$ -plats, Σ_{top} and Σ_{bot} are typically of minimal genus, and that they are non-isotopic in M . Examples of arbitrarily high genus Heegaard splittings which are isotopic after one stabilization were found by Sedgwick (see [19]). However, it is not known whether the Heegaard splittings in those examples are non-isotopic before the stabilization, nor whether that they are stabilizations of a common low genus Heegaard splitting.

Haguiwara (see [5]) has shown that the canonical top and bottom splittings for $2n$ -plats become isotopic after at most $2n - 1$ stabilizations on each of them. We give a short proof of his result in Section 7, as well as fairly general geometric conditions on the plat which ensure that fewer stabilizations suffice (see Proposition 7.2).

We consider the elements of the set $\mathcal{H}(M)$ as vertices of a graph in the plane. The vertices are assembled into horizontal levels according to the genus of the Heegaard splittings. An edge will connect any two vertices (isotopy classes of Heegaard splittings) if one can be obtained from the other by a single stabilization. The graph $\mathcal{H}(M)$ is a 1-ended tree (by a well known result of Reidemeister–Singer), which we call the *Heegaard tree* for M . The results of this paper, as well as all previous results known to us, indicate that $\mathcal{H}(M)$ may in general have the following structure:

There is a finite *root* part of $\mathcal{H}(M)$, which contains all irreducible vertical splittings. Heegaard splittings of the same genus in the root of $\mathcal{H}(M)$ may well need more than one stabilization before they become isotopic, although such a phenomenon has never been proved so far. The maximal level of this root part consists of a single point, and from this point there starts an infinite ray moving upward, called the *trunk* of $\mathcal{H}(M)$. At each vertex level of the trunk, or even of the root part, there may be *branches* attached, i.e., edges which go down into the next lower level. Their lower endpoint (a vertex of $\mathcal{H}(M)$ not on the trunk) represents an irreducible horizontal Heegaard splitting. In all manifolds known to us these branches all have length 1.

Since there are only finitely many isotopy classes of Heegaard splittings of the same genus (by recent results of Pitts–Rubinstein and Stocking [20]), there are only finitely many such branches at

each level. There are examples (see [8]) where the number of these vertices grows polynomially, if one moves up the trunk.

1. Heegaard splittings of pairs

In this section we define the basic set up for this paper. For general definitions and terminology see [3,6,15]. At the end of the section we give a short survey about the development of the notions of vertical and horizontal Heegaard splittings.

A compression body W is a 3-manifold with a preferred boundary component $\partial_+ W$ and is obtained from a collar of $\partial_+ W$ by attaching 2-handles and 3-handles, so that the connected components of $\partial_- W = \partial W - \partial_+ W$ are all distinct from S^2 . The extreme cases, where W is a handlebody, i.e., $\partial_- W = \emptyset$, or where $W = \partial_+ W \times I$, are admitted. Notice that, contrary to the original definition in [4], we require here (as in [17,18]) that compression bodies be connected.

A Heegaard splitting (W_1, W_2) of a 3-manifold M , possibly with non-empty boundary, is a decomposition $M = W_1 \cup W_2$, where the W_i are compression bodies and $W_1 \cap W_2 = \partial_+ W_1 = \partial_+ W_2$. The genus of the Heegaard surface $\Sigma = \partial_+ W_1 = \partial_+ W_2$ is called the genus of the Heegaard splitting.

A Heegaard splitting (W_1, W_2) of a 3-manifold M is called weakly reducible if there are disjoint essential disks $D_1 \subset W_1$ and $D_2 \subset W_2$. Otherwise it is called strongly irreducible. A Heegaard splitting is called reducible if there are two essential disks $D_1 \subset W_1$ and $D_2 \subset W_2$ so that $\partial D_1 = \partial D_2$; otherwise it will be called irreducible.

Given a handlebody H , let $D \subset H$ be a collection of disks so that $H - \mathring{N}(D)$ is a collection of 3-balls. A wave ω with respect to D is an arc in ∂H so that $\partial \omega$ is contained in a component of D , furthermore $\omega \cap D = \emptyset$, and ω meets D from one side and is not homotopic relative $\partial \omega$ into D .

Definition 1.1. Let K be a knot or a link in a 3-manifold M . A *Heegaard splitting of the pair* (M, K) is a Heegaard splitting of M where the Heegaard surface Σ contains K as a union of simple closed curves.

Definition 1.2. A Heegaard splitting for the pair (M, K) is called *vertical* if for each component of K there is some properly embedded essential disk in one of the two handlebodies which is intersected transversally precisely once by the component of K , and it is called *horizontal* if $\Sigma - K$ is incompressible in $M - K$ (which is the same as saying that it is incompressible in both handlebodies).

Notice that if the genus of a vertical Heegaard surface Σ of the pair (M, K) is bigger than the number of components of K then $\Sigma - K$ is always compressible. Hence the vertical and the horizontal case are in this sense extreme opposites of each other.

First examples of a horizontal Heegaard surface Σ for a pair (S^3, K) are given by any incompressible *free* Seifert surface S for the link $K \subset S^3$. Any link in S^3 has a free Seifert surface S , i.e., an orientable surface $S \subset S^3$ with $\partial S = K$ such that the complement of S is a handlebody, and Σ is obtained by simply defining $\Sigma = \partial N(S)$. The Seifert algorithm for obtaining a Seifert surface for a knot or link always gives such a free Seifert surface. If S is incompressible, then Σ

will be horizontal. However, in general it is not true that a free Seifert surface will be incompressible. In fact, there are knots in S^3 for which any free Seifert surface must be compressible [12].

If Σ is a vertical Heegaard surface for the pair (M, K) then it gives rise to a Heegaard splitting for the manifold $M - N(K)$. This splitting is obtained by isotoping each component K_i off Σ into the handlebody which contains the disk punctured once by K_i . The handlebodies H_1 and H_2 are then transformed into compression bodies W_1 and W_2 in $M - N(K)$ which together determine a Heegaard splitting for $M - N(K)$. In particular, since the components K_i are core curves of the original handlebodies, this gives a Heegaard splitting for all closed manifolds obtained by surgery on K , for any surgery value.

If, on the other hand, the Heegaard surface Σ of the pair (M, K) is not vertical, it will in general not be isotopic in M to a Heegaard surface for $M - N(K)$. The boundary of a neighborhood $N(K_i)$ of each component K_i of K , $i = 1, \dots, d$, is cut by Σ into two annuli A_i^1 and A_i^2 . The surface $\Sigma - N(K)$ determines a splitting (not a Heegaard splitting!) of $M - N(K)$ into two handlebodies W_1 and W_2 which are glued along $\Sigma - N(K) = \partial W_1 - \cup A_i^1 = \partial W_2 - \cup A_i^2$. If for each i we glue the two annuli A_i^1 and A_i^2 together by a multiple Dehn twist along either of them, the Heegaard surface Σ will define a Heegaard splitting of the resulting manifold.

Let $\beta_i \subset \partial N(K_i)$ be a curve dual to a component of $\partial \Sigma \cap \partial N(K_i)$ denoted by $\partial_i \Sigma$, i.e., a curve on $\partial N(K_i)$ intersecting $\partial_i \Sigma$ in a single point. We can choose β_i to bound a meridian disk in $N(K_i)$. Then, for any integer k_i , gluing the annuli A_i^1 and A_i^2 together via a k_i -fold Dehn twist is equivalent to performing $\frac{1}{k_i}$ -surgery on K_i with respect to the basis $(\beta_i, \partial_i \Sigma)$ for $H_1(\partial N(K_i))$, and conversely. Let us denote by $\Sigma(\frac{1}{\hat{k}})$, where $\hat{k} = (k_1, \dots, k_d)$, the manifold obtained by $\frac{1}{k_i}$ -surgery on each K_i . Hence for all pairs (M, K) the surface Σ determines a Heegaard splitting for the pair $(\Sigma(\frac{1}{\hat{k}}), K)$.

In unpublished work Casson and Gordon proved the following important result (see [14] for a proof). It is formulated here in the terminology introduced above:

Theorem 1.3 (Casson–Gordon). *Let $K \subset S^3$ be a knot and Σ a horizontal Heegaard surface for the pair (S^3, K) . Then for all manifolds $\Sigma(\frac{1}{\hat{k}})$, with $|\hat{k}| \geq 6$, the Heegaard splitting determined by the surface Σ is strongly irreducible.*

If $K \subset M$ is a link in some manifold M and Σ is a Heegaard surface for M , then we say that Σ is *vertical* (or *horizontal*) *with respect to K* if K can be isotoped onto Σ to give a vertical (or horizontal) Heegaard splitting of the pair (M, K) . If the reference to K is unambiguous, we sometimes simply say that Σ is vertical (or horizontal).

We conclude this section by giving some of the history of vertical and horizontal Heegaard splittings. None of the following is used in the sections to come.

Vertical Heegaard splittings were first defined by Boileau and Otal in the context of Seifert fibered spaces over S^2 with three exceptional fibers. These are Heegaard splittings where the handlebodies contain the exceptional fibers as cores, i.e., as curves which meet an essential disk in one of the handlebodies in a single point. It was known already then, by an observation of Casson and Gordon, that there were other Heegaard splittings for these Seifert fibered spaces and that one could isotope an exceptional fiber onto the Heegaard surface also in these cases [1]. The Heegaard splittings for general Seifert fibered spaces that were described by Boileau and Zieschang in [2] before the work of Boileau–Otal were by our definition all vertical with respect to any of the

exceptional fibers, while the exceptional Heegaard splittings, case (i) of Theorem 1.1 of [2], are Heegaard splittings of the pair (M, f) where f is an exceptional fiber.

In unpublished work Casson and Gordon showed that one could find more examples of horizontal Heegaard splittings. They showed that some of the manifolds obtained by surgery on certain generalized pretzel knots admit irreducible Heegaard splittings, where the core curve of the surgery torus can be isotoped onto the Heegaard surface. The complementary surface is incompressible to both sides, thus defining a horizontal splitting of the pair. These knots are all hyperbolic knots [7], which shows that horizontal Heegaard splittings are not confined to Seifert fibered spaces.

The viewpoint that these exceptional Heegaard splittings were in fact not an exotic phenomenon at all was strengthened by a structure theorem for irreducible Heegaard splittings of negatively curved 3-manifolds, proved by the second author and Rubinstein [13]. They showed that, given a link in a negatively curved 3-manifold and the full collection of manifolds obtained by surgery on this link, then *all* Heegaard surfaces for “almost all” of these manifolds come from Heegaard surfaces of the pair (M, K) .

Horizontal Heegaard splittings were, in the case of general Seifert fibered spaces, introduced first by the second author and Jennifer Schultens [14]. They showed that for orientable Seifert fibered spaces all Heegaard splittings are stabilizations of either vertical or horizontal Heegaard splittings. Here a horizontal Heegaard splitting (see Definition 3.1 of [14]) of a Seifert fibered space is one which is obtained from a surface fibration over the circle of the complement, in the manifold, of a fiber. Note that in this case the fiber can be isotoped onto the Heegaard surface and that the Heegaard surface less a neighborhood of the fiber is incompressible in both handlebodies. Note also that not all Seifert fibered spaces have such Heegaard splittings.

2. Trellis Heegaard splittings

Let T be a graph in a vertical plane $P \subset \mathbb{R}^3$ which consists of horizontal and vertical edges only. Every maximal connected union of horizontal edges of T is called a *horizontal line*. The union of two adjacent horizontal lines and all vertical edges spanned between them is called a *horizontal layer*. If T has m horizontal layers and contains n vertical edges in each of them, arranged in “brick like fashion” as in Fig. 1, it is called a *standard trellis* of size (m, n) . Its regular neighborhood in \mathbb{R}^3 is a handlebody $H_1 = N(T)$ of genus $g(T) = m(n - 1)$, embedded in the standard way in S^3 , which we identify with the one-point compactification of \mathbb{R}^3 .

For any integer $(m \times n)$ -matrix $A = (a_{i,j})$ we define a knot or link $K = K(A)$ contained in the boundary of the handlebody H_1 and winding around the trellis T as in Fig. 2. There each configuration as in Fig. 3, occurring at the j th twist box, counted from the left, of the i th layer, counted from the top, indicates $a_{i,j}$ half twists. We call $a_{i,j}$ the *twist coefficients* and A the *twist matrix*. We always use P as the projection plane for $K = K(A)$.

Note that the long horizontal strings are on the back of the trellis.

Whenever a trellis $T \subset S^3$ and a knot or link $K \subset S^3$ are given, and K is isotopic to $K(A)$ for some twist matrix A as above, then we say that K is *carried by T with twist matrix A* .

As the complement $H_2 = S^3 - N(T)$ of H_1 is also a handlebody, the pair (H_1, H_2) defines a Heegaard splitting of the pair (S^3, K) , which we call the *trellis Heegaard splitting*. We refer to

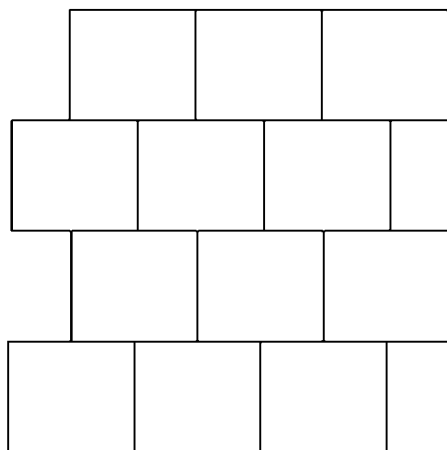


Fig. 1.

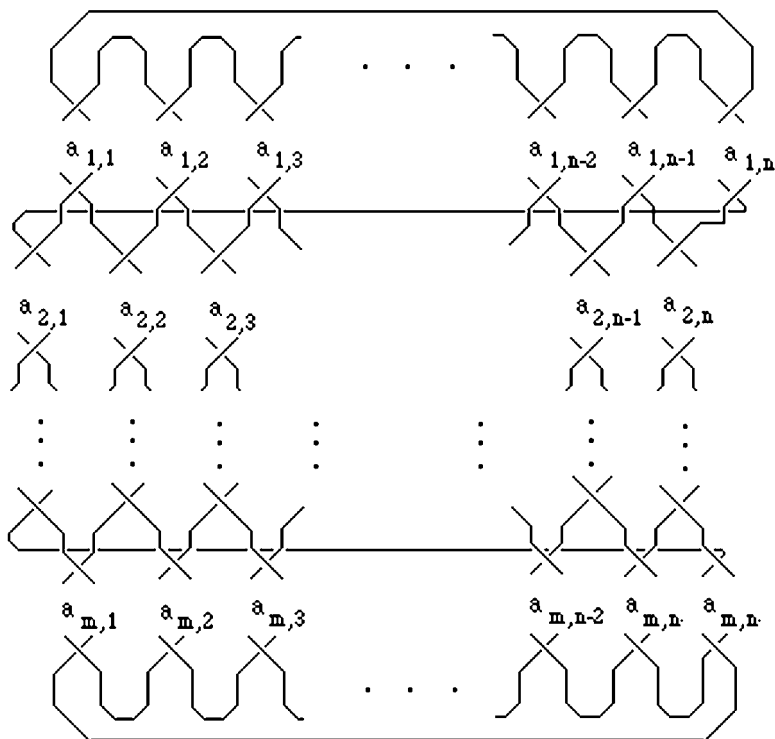


Fig. 2.

H_1 as the *inner* handlebody and to H_2 as the *outer* handlebody of this splitting. As in the last section we denote the surface which is their common boundary by Σ . The plane P cuts Σ into two connected components which we refer to as the *front* and the *back*.

Notice that K bounds a possibly non-orientable surface S in H_1 , defined by replacing every vertex of T by a small disc in P and every edge of T by a twisted band attached to those disks.

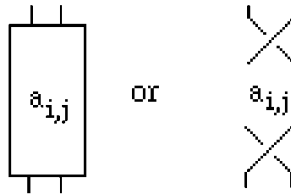


Fig. 3.

Lemma 2.1. *The surface $\Sigma - K$ is incompressible in the inner handlebody H_1 .*

Proof. The handlebody H_1 admits a structure of an orientable I -bundle over the surface S . Hence $\Sigma - K$ is isotopic in ∂H_1 to the induced orientable ∂I -bundle over S , which is the orientable double cover of S in case S is unorientable, or, if S is orientable, it is the disjoint union of two copies of S . In particular the fundamental group of (a component of) $\Sigma - K$ is mapped injectively to $\pi_1(S) = \pi_1(H_1)$. \square

Lemma 2.2. *If $n \geq 3$ and if all twist coefficients satisfy $|a_{i,j}| \geq 3$ then $\Sigma - K$ is incompressible in the outer handlebody H_2 .*

Proof. Notice that the projection plane P cuts the handlebody H_1 through the middle. Let $\mathcal{D} = \{D_{i,j}\}$, for $1 \leq i \leq m$ and $1 \leq j \leq n-1$, be the set of disks given by those connected components of $P \cap H_2$ which are compact. The complement of \mathcal{D} in H_2 is a 3-ball.

We now want to remove all inessential intersections of K with \mathcal{D} by an isotopy of K on Σ (“tightening K with respect to \mathcal{D} ”). Such inessential intersections occur only at the top or the bottom horizontal line of T . It occurs exactly if some $a_{1,j}$ is negative and $a_{1,j+1}$ is positive, or if $a_{m,j}$ is positive and $a_{m,j+1}$ is negative. Hence by isotoping some of the top and some of the bottom arcs of $K - P$ from the front to the back of Σ we eliminate all of the inessential intersections. Notice that our assumption $|a_{i,j}| \geq 3$ implies that after these isotopies each *vertical column* of H_1 , i.e., the neighborhood of a vertical edge of the underlying trellis, has at least one small horizontal arc of K on the front of Σ which connects the two adjacent disks $D_{i,j}$ and $D_{i,j+1}$, and another such arc on the back.

Let γ be a loop in $\Sigma - K$ which is contractible in H_2 and transverse to \mathcal{D} . Hence, after tightening γ with respect to \mathcal{D} , the loop γ either misses \mathcal{D} or else it contains a wave with respect to \mathcal{D} (see Section 1).

It follows from our assumption $|a_{i,j}| \geq 3$ that the connected components Σ_i of $\Sigma - \mathring{N}(K - \mathcal{D})$ do not contain essential loops. Therefore the curve γ must meet \mathcal{D} and hence it must contain a wave γ_0 .

This wave γ_0 is a properly embedded arc in some connected component Σ_i of $\Sigma - \mathring{N}(K - \mathcal{D})$. Its endpoints are on some $\partial D_{i,j}$, (more precisely, on the parallel copies of $\partial D_{i,j}$ on $\partial N(D_{i,j})$), and γ_0 must join two distinct connected components of the intersection $D_{i,j} \cap \Sigma_i$ from the same side of $D_{i,j}$. Thus we have reduced our goal, showing that there is no essential loop in $\Sigma - K$ which bounds a disc in H_2 , to showing the following:

Claim. For each connected component Σ_i of $\Sigma - \overset{\circ}{N}(K - \mathcal{D})$ the intersection of Σ_i with any $D_{i,j}$ is either empty, or consists of precisely one arc, or consists of precisely two arcs along which Σ_i meets $D_{i,j}$ from opposite sides of P .

To prove this claim we divide the complementary components of $K \cup \mathcal{D}$ in Σ into finitely many classes, according to their position on Σ as pictured in Fig. 4.

Those complementary components which are just small horizontal strips on the front or the back of a vertical column of H_1 satisfy the claim, as they meet \mathcal{D} in precisely two arcs, which belong to distinct $D_{i,j}$ unless the column is an outermost one. For the outermost columns the horizontal strips run around from the front to the back and hit the same disk $D_{i,j}$ twice, but from opposite sides of P , so they also satisfy the claim.

Next we consider the class of complementary regions Σ_i which are located on the front of Σ , and are in one to one correspondence with the valence-3-vertices of T other than those on the top or on the bottom horizontal line. Each such “triangular shaped” Σ_i can meet at most four disks $D_{i,j}$, and these are all distinct, unless Σ_i is outermost on its horizontal layer. In the latter case we notice that the assumption $n \geq 3$ implies that the triangular region cannot be outermost simultaneously to the right and to the left. Hence at most two of the four intersection arcs may belong to the same disk $D_{i,j}$, but then Σ_i meets that disk $D_{i,j}$ from opposite sides of P , which proves the claim for this class of regions as well.

A third class of complementary components Σ_i arises on the back of Σ . Each such Σ_i contains in its boundary one of the subarcs of K which have been isotoped from the front to the back in the tightening process of K , and Σ_i meets at most three distinct $D_{i,j}$. If Σ_i is outermost on its horizontal layer two of the $D_{i,j}$ will agree, but then they are met by Σ_i from opposite sides of P .

It remains to check the last class, consisting of “long horizontal” complementary regions, one on the top front of the first layer, one on the bottom front of the last layer, and two regions on the back of each layer. However, it is easy to check that each of those regions meets any non-outermost (in its horizontal level) disc $D_{i,j}$ in at most one arc, while the outermost discs $D_{i,j}$ could be met by some regions possibly twice, but if that happens then they are met from opposite sides of P . This proves the Claim and hence the lemma. \square

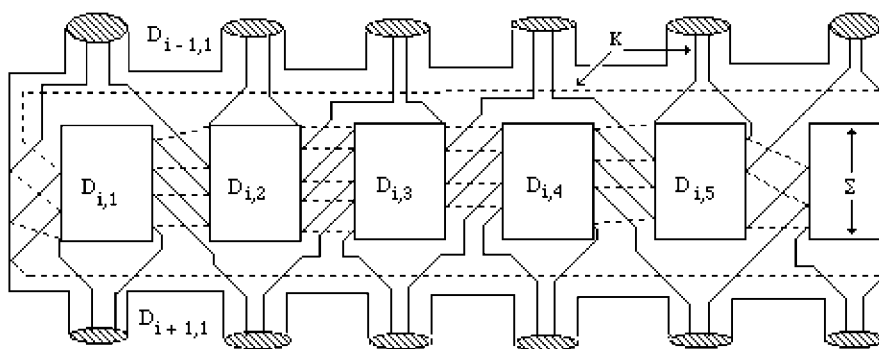


Fig. 4.

In what follows we will admit more general knots $K = K(A)$, than the ones considered so far: We start with a standard trellis T of size $m \times n$ and remove any number of vertical edges or horizontal edges, with the following restrictions: There are at least three vertical edges in each layer, there is only one horizontal line in each horizontal level, there are no vertices of valence one, and the trellis is connected. The resulting graph T' will be called a *generalized trellis*. As before, a knot or link carried by the generalized trellis defines a twist matrix A , which is an $(m \times n)$ -matrix A with integer coefficients, except that we use the convention that we set $a_{i,j} = \infty$ for those entries of A which correspond to the vertical edges $e_{i,j}$ of T that were deleted when passing over to T' .

Conversely, given such a matrix A , the knot or link $K = K(A)$ is built in the neighborhood of the deleted edges on the local model used for standard trellisses at the top and at the bottom horizontal lines, so that all the terminology and all the basic facts for standard trellisses extend to the case of a generalized trellis T as well. We define the *genus* $g(T)$ of T to be the genus of the handlebody $H_1 = N(T)$.

The proof for the incompressibility of $\Sigma - K$ in the inner handlebody H_1 (as in Lemma 2.1) carries over word by word to generalized trellisses. In order to prove the incompressibility of $\Sigma - K$ in the outer handlebody H_2 we need to make the following adjustments in the proof of Lemma 2.2:

- (a) In the tightening process of K with respect to D , we may need to isotope additional arcs from the front to the back. These additional arcs will occur at the top or the bottom of the deleted vertical edges.
- (b) We will now have to consider regions which replace the triangular shaped components of $\Sigma - N(K - D)$ on the front of Σ for a standard trellis, but are more complicated. These regions may now have more “sides”: They do not necessarily correspond any more to single vertices on interior horizontal lines of T , but rather to segments on such lines. These segments contain only vertices which bound vertical edges from above or only vertical edges from below, and are maximal with respect to this property. It is easy to see that these new regions still satisfy the Claim in the proof of Lemma 2.2, so that the proof of this lemma carries over directly to generalized trellisses T .

We summarize the results of this section with the following:

Proposition 2.3. *Let $K = K(A)$ be a knot or link carried by a generalized trellis T with twist matrix A that has coefficients $a_{i,j} \in \mathbb{Z} \cup \{\infty\}$. If all twist coefficients satisfy $|a_{i,j}| \geq 3$ then the trellis Heegaard splitting of the pair (S^3, K) associated to T is horizontal. \square*

The proofs of Lemma 2.2 and Proposition 2.3 show that the condition $|a_{i,j}| \geq 3$ is by no means a necessary condition for both statements. For example, a local necessary condition is that not both of $a_{i,j}$ and $a_{i,k}$ be 0 for $j \neq k$. However, it seems difficult, at this stage, to give precise necessary and sufficient conditions.

Definition 2.4. For any knot K carried by a generalized trellis T we define the *trellis linking number* $a(K)$ as the element of $H_1(S^3 - N(K))$ determined by a component of $\partial(\Sigma - N(K))$, where $\Sigma = \partial N(T)$.

The trellis linking number $a(K)$ can be computed as follows: Choose an orientation for K . Let A be the twist matrix of K . Define A_0 to be the set of all twist coefficients $a_{i,j} \in A$ with the property

that the two oriented strings of the knot K cross through the corresponding twist box of the trellis projection in the same vertical direction. Notice that in this case the local linking of the knot with a parallel curve on the surface Σ is twice the twist coefficient $a_{i,j}$. If the orientations of the strings are opposite then the linking number is 0. Notice also that the strings of K on the back of the trellis do not contribute to the local linking. Hence $a(K) = 2\sum\{a_{i,j} | a_{i,j} \in A_0\}$. In particular the boundary slope on $\partial N(K)$ determined by $\partial(\Sigma - N(K))$, expressed in the usual meridian/longitude coordinates of $H_1(\partial N(K))$, is $\frac{a(K)}{1}$. It follows that $\Sigma\left(\frac{1}{k}\right) = K\left(\frac{1 + ka(K)}{k}\right)$.

3. Flypes

Let T be a generalized trellis. We say that two adjacent vertical edges $e_{i,j}, e_{i,k}$ in the i th interior horizontal layer, with $i \neq 1, m$, is an *interior pair of edges* if neither $e_{i,j}$ nor $e_{i,k}$ is outermost, and if the segments of the two horizontal lines bounded by the vertices of $e_{i,j}$ and $e_{i,k}$ satisfy the following properties:

- There are no vertical edges in the $(i - 1)$ th or in the $(i + 1)$ th horizontal layer which have endpoints on either of the two segments.
- There are two vertical edges in the $(i - 1)$ th and two in the $(i + 1)$ th horizontal layer which have endpoints separating the two segments from the endpoints of all other vertical edges in the i th layer. This is illustrated in Fig. 5.

A *flype* at the interior pair of edges $(e_{i,j}, e_{i,k})$ is an ambient isotopy of K which is obtained as follows: Consider a box in S^3 which intersects K in exactly the two subarcs of K winding around the edge $e_{i,j}$, the two subarcs winding around $e_{i,k}$, and in the two horizontal subarcs on the front of Σ connecting the top of $e_{i,j}$ to the top of $e_{i,k}$, and the bottom of $e_{i,j}$ to the bottom of $e_{i,k}$ respectively (see Fig. 6).

A flype will flip the box by 180° about a horizontal axis leaving all parts of the knot outside the box fixed. This operation changes the projection of K in P by adding a crossing on the left and a crossing on the right side of the box. These crossings have opposite signs.

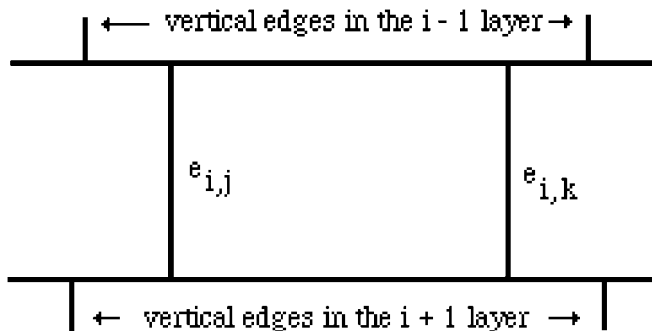


Fig. 5.

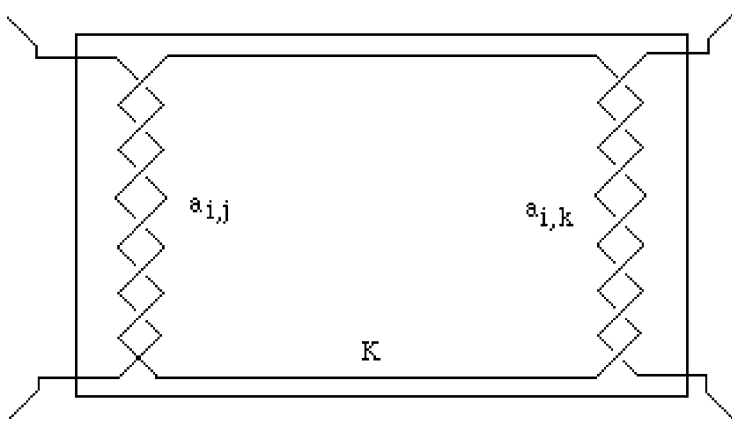


Fig. 6.

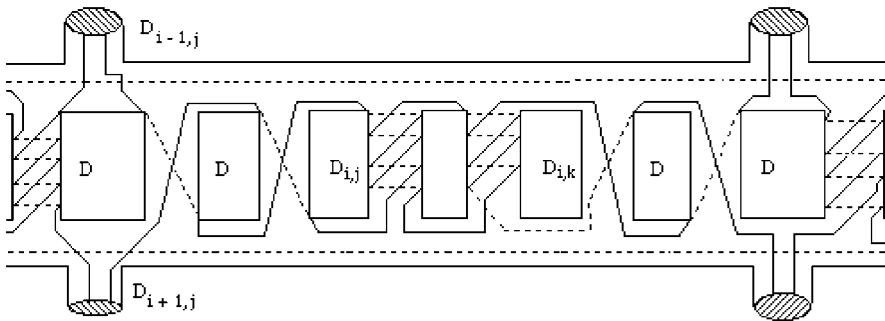


Fig. 7.

The projection of K obtained after a flype is carried by a new trellis. It differs from T in that there is a new vertical edge on the left side of $e_{i,j}$ and another new one on the right side of $e_{i,k}$, one with twist coefficient 1 and the other one with -1 . The flype will be called *positive* if the coefficient of the right “new” edge is positive. A positive/negative flype iterated $\pm n$ times will be called an $\pm n$ -flype (see Fig. 7). When the interior pair of edges in which the n -flype is performed is specified we will denote the image of K after the n -flype by $K(n)$ and the new trellis with the new $2n$ vertical edges by $T(n)$. Similarly we will denote $N(T(n))$ by $H_1(n)$ and $\partial H_1(n)$ by $\Sigma(n)$.

As before, the inner handlebody $H_1(n)$ is cut by the projection plane P through the middle, and the compact components of the intersection of $H_2(n) = S^3 - H_1(n)$ with P give a collection $\mathcal{D}(n)$ of disks, which cut $H_2(n)$ open to give a 3-ball. The disk collection $\mathcal{D}(n)$ consists precisely of the disks $D_{i,j}$ defined as in the last section for T , and, for each flype, an additional two disks, one on the left of $e_{i,j}$ and one on the right of $e_{i,k}$.

Our next goal is to show that the surface $\Sigma(n) - K(n)$ is incompressible in $H_2(n)$. As in the last section, we first have to tighten $K(n)$ with respect to the disk system $\mathcal{D}(n)$. This is done by moving some arcs from the front to the back part of $\Sigma(n)$, as explained in the last section for generalized

trellisses. In this tightening procedure we first isotope those arcs from the front of $\Sigma(n)$ to the back which already had to be moved in order to tighten K with respect to \mathcal{D} . The only place where $K(n)$ may not be tight, after these “old” tightening isotopies, are horizontal arcs with one endpoint on the top or on the bottom of the vertical column corresponding to the edges $e_{i,j}$ or $e_{i,k}$. This is because all new left vertical edges have the same sign for their twist coefficient, and similarly for all new right edges (with opposite sign). There are various cases according to the sign of the twist coefficients $a_{i,j}$ and $a_{i,k}$, and the sign of the flype, and they will be discussed in the proof of Lemma 3.2 below.

If we try to show the incompressibility of $\Sigma(n) - K(n)$ in the outer handlebody as before we will quickly run into a problem, as it will turn out that often the disk system $\mathcal{D}(n)$ decomposes $\Sigma(n) - K(n)$ into subsurfaces and some of them do *indeed* contain a wave. Thus we first need to generalize our method:

For any knot or link $K \subset \Sigma$ and a disk system \mathcal{D} which cuts H_2 into one (or more) 3-balls let us consider, as before, a decomposition of Σ into subsurfaces Σ_i which are simply connected and which have boundary on $K \cup \partial\mathcal{D}$. We require as before that Σ_i meets K only in proper subarcs of $\partial\Sigma_i$, but contrary to the above we allow the possibility that Σ_i contains some properly embedded arcs from $\partial\mathcal{D}$. In other words, the decomposition considered here arises from the connected components of $\Sigma - (K \cup D)$ by gluing together some of these components along segments of $\partial\mathcal{D}$.

Let γ be a path which is properly embedded in Σ_i and transverse to \mathcal{D} , with boundary points on two distinct components of $\partial\Sigma_i \cap \mathcal{D}$. Notice that, as Σ_i is simply connected, up to a homotopy of $(\gamma, \partial\gamma)$ in $(\Sigma, \partial\Sigma - K)$ there are only finitely many such paths. We read off the word corresponding to the intersections of $\hat{\gamma}$ with the disks from \mathcal{D} , and freely reduce it to get the *interior word* $w(\hat{\gamma})$. Let $w(\gamma)$ be the analogously defined word, but with the two extra intersections of γ with \mathcal{D} at the two boundary points of γ . These words are invariant modulo free reduction, with respect to relative homotopy of γ . As Σ_i is simply connected these words only depend on the two components of $\partial\Sigma_i \cap \mathcal{D}$ which contain the endpoints of γ .

Lemma 3.1. *If for each such γ the freely reduced words $w(\gamma)$ and $w(\hat{\gamma})$ satisfy the equation*

$$\text{length}(w(\gamma)) = \text{length}(w(\hat{\gamma})) + 2,$$

then $\Sigma - K$ is incompressible in the outer handlebody H_2 .

Proof. Every loop ρ in $\Sigma - K$, after being made transverse and tight with respect to \mathcal{D} , decomposes into arcs γ_i as above, which are concatenated along their boundary points: $\rho = \gamma_1 \gamma_2 \dots \gamma_q$. By assumption any letter which corresponds to one of these boundary points, say $\gamma_i \cap \gamma_{i+1}$ (with i understood mod q), does not cancel with either of the adjacent reduced interior words $w(\hat{\gamma}_i)$ or $w(\hat{\gamma}_{i+1})$ (or against the first letter coming from the next arc γ_i , in case the interior word is empty).

Hence the whole loop reads off a reduced word which is non-trivial, and thus it can not be contractible in H_2 . \square

Notice that if no Σ_i contains any properly embedded arc from $\partial\mathcal{D}$ then Lemma 3.1 coincides with the old criterion that no Σ_i may contain a wave.

This lemma will be applied below in a particular situation, which we want to spell out explicitly. It is easy to see that in this situation the hypotheses of Lemma 3.1 are satisfied for the regions $\bar{\Sigma}_i$ defined below.

Lemma 3.2. *Assume that one has a decomposition of Σ along $K \cup \mathcal{D}$ as before. Assume also that for each wave γ_0 in any of the components Σ_i the two regions Σ_j, Σ_k adjacent to Σ_i which contain the endpoints of γ_0 satisfy the following conditions:*

- (a) Σ_j and Σ_k do not contain waves,
- (b) Σ_j and Σ_k do not meet any of the curves $\partial D_{r,s} \in \partial D$ from the same side, and
- (c) the union $\bar{\Sigma}_i$ of Σ_i with all adjacent regions which contain an endpoint of a wave in Σ_i is simply connected.

Then K is incompressible in the outer handlebody H_2 . \square

We are now ready to prove:

Lemma 3.3. *Let $K = K(A)$ be a knot or link carried by a generalized trellis T . Assume that all coefficients $a_{i,j}$ of A which are different from ∞ satisfy $|a_{i,j}| \geq 3$, and that there is an interior pair of edges $(e_{i,j}, e_{i,k})$ of T with $|a_{i,j}|, |a_{i,k}| \geq 4$. Then for any n -flype at $(e_{i,j}, e_{i,k})$ the surface $\Sigma(n) - K(n)$ is incompressible in $H_2(n)$.*

Proof. As the flype involves only a local part of the trellis and the knot or link carried by it, we can use the fact shown in the proof of Lemma 2.2 that those components $\Sigma_i(n)$ of $\Sigma(n) - (K(n) \cup \partial \mathcal{D}(n))$ which have not been changed by the flype do not contain a wave. Thus it suffices to investigate those “new” components $\Sigma_i(n)$ which intersect the flype box defined above. We will have to distinguish various cases according to the sign of the flype number n and of the twist coefficients $a_{i,j}$ and $a_{i,k}$. In each of these cases there will be the following types of “new” complementary components of $K(n) \cup \mathcal{D}(n)$ in $\Sigma(n)$:

- (a) Small “horizontal” strips on the front or on the back of the vertical columns corresponding to $e_{i,j}$ and $e_{i,k}$.
- (b) Two long horizontal regions on the front, which bound all of the new disks: One of the long regions bounds from above and the other long region from below.
- (c) Two similarly long horizontal regions on the back.
- (d) Regions on the back which bound one of the arcs of $K(n)$ moved to the back by our tightening isotopies above, and which bound precisely three disjoint disks from $D(n)$.

We now distinguish the following three cases, the third of which is pictured in Fig. 7. All other possibilities can be treated similarly to one of them, by the two mirror-symmetries of the given set up.

- I. $n \leq 1$ and $a_{i,j} \geq 4$ and $a_{i,k} \leq -4$,
- II. $n \leq 1$ and $a_{i,j} \leq -4$ and $a_{i,k} \geq 4$,
- III. $n \leq 1$ and $a_{i,j} \geq 4$ and $a_{i,k} \geq 4$.

In cases I and II there is precisely one region of type (d), and in case (III) there is none. In any case, such regions never contain a wave. Clearly the regions of type (a) or (d) never contain a wave.

In case I we check from Fig. 7 that none of the four long horizontal regions of type (b) or (c) contains a wave. In case II there are two such regions with precisely one wave each, namely the bottom region of type (b), and the top region of type (c). The other two long horizontal regions do not contain waves. Similarly, in case III there are two long horizontal regions which contain a wave: The bottom region of type (b), and the top region of type (c).

Observe that in each case the two adjacent regions which contain the endpoints of the wave γ are always of type (a), and the two never belong to the same vertical column. It is easy to check that the conditions of Lemma 3.2 are satisfied, which proves that the surface $\Sigma(n) - K(n)$ is incompressible in $H_2(n)$. \square

Theorem 3.4. *Let T be a generalized trellis and let $K = K(A) \subset S^3$ be a knot or link carried by T with twist matrix A . Assume that all coefficients $a_{i,j}$ of A which are different from ∞ satisfy $|a_{i,j}| \geq 3$ and that there is an interior pair of edges $(e_{i,j}, e_{i,h})$ of T with twist coefficients $|a_{i,j}|, |a_{i,h}| \geq 4$. Then for all $n \in \mathbb{Z}$ the trellis $T(n)$ obtained from an n -flype at this edge pair defines a trellis Heegaard splitting for the pair (S^3, K) which is horizontal and of genus $g(T(n)) = g(T) + 2n$.*

In particular, if K is a knot, then for all the manifolds $K((1 + ka(K))/k)$ with $|k| \geq 6$ this induces a strongly irreducible Heegaard splitting of genus $g(T) + 2n$, for all $n \in \mathbb{Z}$.

Proof. For all $n \in \mathbb{Z}$ the surface $\Sigma(n) - K(n)$ is incompressible in the inner handlebody H_1 by Lemma 2.1, and it is incompressible in the outer handlebody H_2 by Lemma 3.3. Hence the trellis splitting defined by $T(n)$ is horizontal, and as a consequence of Casson–Gordon’s result, stated in Theorem 1.3, this gives a strongly irreducible Heegaard splitting of genus $g(T) + 2n$ for the surgery manifolds $K((1 + ka(K))/k)$, with $|k| \geq 6$. \square

Proof of Theorem 0.2. The theorem follows directly from Theorem 3.4. \square

4. Horizontal Heegaard splittings for knots in plat projections

In this section we apply the tools developed in the previous two sections for knots or links carried by a trellis, to knots or links given as plats (see [3] and Fig. 8).

A $2n$ -plat projection as above, determines a $(m \times n)$ -prematrix \hat{A} with integer twist coefficients $a_{i,j}$. A $(m \times n)$ -prematrix is a $(m \times n)$ -“matrix” where the odd numbered rows have only $n - 1$ entries instead of n . Precisely, for i odd we have $1 \leq j \leq n - 1$, while for i even we have $1 \leq j \leq n$.

A prematrix \hat{A} determines, in a canonical way, a matrix A by defining $a_{i,n} = 0$ for all odd indices i . We will say that A is obtained from \hat{A} by 0-filling. A first observation is the following:

Lemma 4.1. *Every knot or link K given as $2n$ -plat with twist prematrix \hat{A} is isotopic to the knot or link $K(A)$ carried by a standard trellis of size (m, n) , with twist matrix A obtained by 0-filling from \hat{A} .*

Proof. For every odd layer of the plat projection one takes the left-most vertical subarc k of K and moves it by an ambient isotopy along the back of the plat projection until it is in a position right of

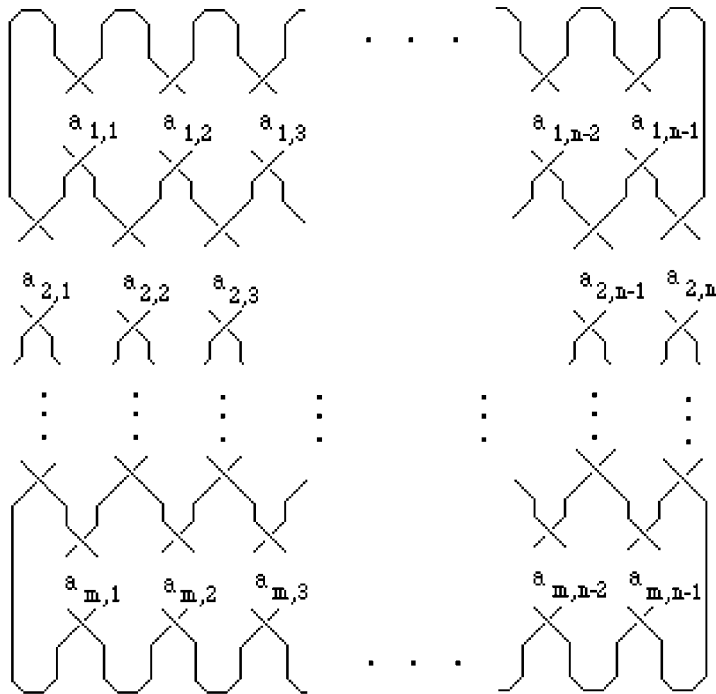


Fig. 8.

the former right-most vertical subarc arc in this layer. This isotopy creates two long horizontal subarcs on the back, connecting the top end point in the old position of the arc k to the top of the new position, and similarly at the bottom of k . We now interpret the two right-most parallel vertical strings of this new projection of K as the n th twist box of this layer (with twist coefficient equal to 0), and observe that this gives a knot or link $K(A)$ precisely as claimed (see Fig. 9). \square

Proposition 4.2. *Let K be a knot or link in a $2n$ -plat projection, let \hat{A} be the associated twist prematrix, and assume that all twist coefficients satisfy $|a_{i,j}| \geq 3$. Then the trellis Heegaard splitting of the pair $(S^3, K(A))$ is horizontal, where the twist matrix A is obtained by 0-filling from \hat{A} .*

Proof. Let T be the standard (m, n) -trellis which carries $K(A)$, and let Σ be the associated trellis Heegaard surface. By Lemma 2.1 the subsurface $\Sigma - K$ is incompressible in $H_1 = N(T)$. Thus it remains to show that $\Sigma - K$ is incompressible in $H_2 = S^3 - \bar{H}_1$. The proof uses the same technique as that of Lemma 2.2.

From the assumption that all twist coefficients of the prematrix \hat{A} associated to the $2n$ -plat K satisfy $|a_{i,j}| \geq 3$ it follows that the twist matrix A for the trellis T satisfies the same condition, except that $a_{i,n} = 0$ whenever i is odd. As in the proof of Lemma 2.2 we consider the decomposition of Σ into subsurfaces Σ_i by cutting along $K \cup D$, where D is the disk system in H_2 considered there. It is shown there that, if all twist coefficients of A satisfy $|a_{i,j}| \geq 3$, then none of the subsurfaces

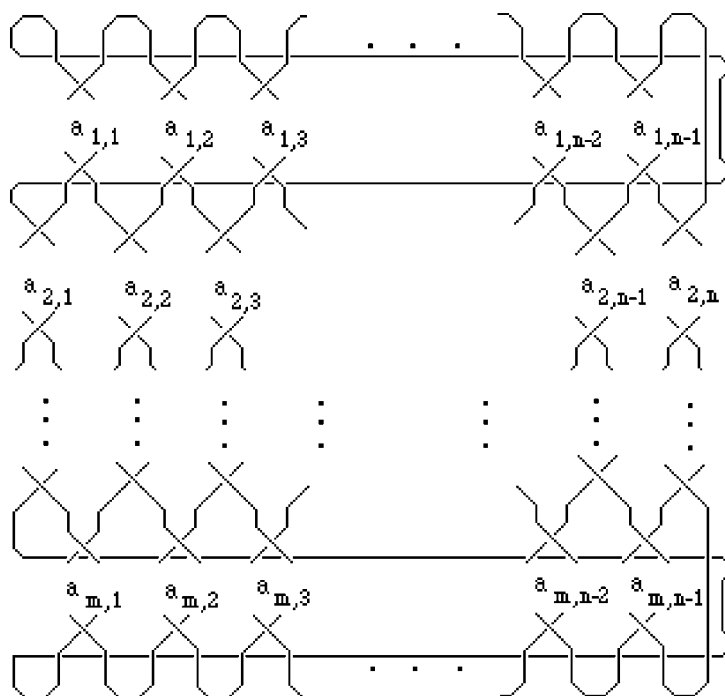


Fig. 9.

Σ_i contains a wave. Hence it suffices to consider only those subsurfaces which meet the right-most vertical column of an odd horizontal layer.

It is easy to see that there are exactly three such complementary regions, and that the two of them which intersect this vertical column only on the front do not contain a wave. However, the third one does contain waves on the back of Σ . For each disk $D_{i,j} \in D$ in this layer, except for the right-most, there is a wave. It starts at the top of $D_{i,j}$, runs horizontally to the right, then down over the right-most vertical column, and then horizontally back to the bottom of $D_{i,j}$. Its two endpoints are in different connected components of $\partial D_{i,j} - K$. A picture is given in Fig. 10.

Notice that the waves pointed out above are the only waves in this region. Hence we can easily verify that the hypotheses of Lemma 3.2 are satisfied which implies the incompressibility of $\Sigma - K$ in H_2 . \square

Given a knot K is in a $2n$ -plat projection we can compute, as in Definition 2.4, $a(K)$ with respect to the standard trellis T given by Lemma 4.1. It is the linking number of a boundary component of the corresponding surface $\Sigma - N(K)$ with K . In this case we will call $a(K)$ the *plat linking number*.

Proof of Theorem 0.1. The surface $\Sigma - N(K)$ is incompressible in the inner handlebody H_1 by Lemma 2.1, and it is incompressible in the outer handlebody H_2 , by Proposition 4.2. Hence the trellis splitting defined by T is horizontal, and as a consequence of Theorem 1.3 this gives a strongly irreducible Heegaard splitting of genus $m(n-1)$ for all surgery manifolds $K((1+ka(K))/k)$, with $|k| \geq 6$. \square

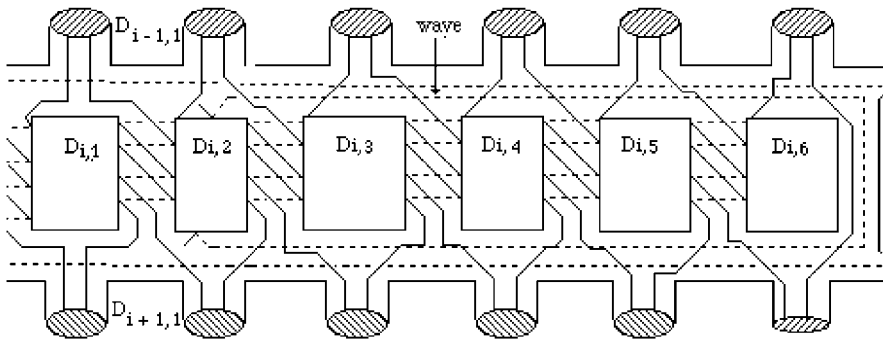


Fig. 10.

5. Canonical Heegaard splittings

The goal of this section is to extend the notion of canonical top and bottom Heegaard splittings which are defined for plat projections of knots or links K , reviewed below, to arbitrary projections of K .

Definition 5.1. Let K be a knot or link given as $2n$ -plat in $\mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$. Let $\tau_1, \dots, \tau_{n-1}$ be a system of pairwise disjoint horizontal arcs which connect adjacent local maxima (*the top bridges*) of K (see Fig. 11). One defines the compression body W_1 to be the union of a collar of $\partial N(K)$ in $S^3 - N(K)$, and of a regular neighborhood of the $\tau_1, \dots, \tau_{n-1}$. The handlebody $H_2 = W_2$ is defined as the complement $(S^3 - N(K)) - W_1$, and together they define the *canonical top Heegaard splitting* of $S^3 - N(K)$ associated to the given $2n$ -plat projection of K . Analogously, if we replace the arcs τ_i by similar arcs $\rho_1, \dots, \rho_{n-1}$ connecting adjacent local minima of K we obtain the *canonical bottom Heegaard splitting* of $S^3 - N(K)$.

Remark 5.2. As both canonical Heegaard splittings are obtained by adding tunnels to a regular neighborhood of K both splittings are (after pushing K on $\partial N(K)$) vertical splittings for the pair (S^3, K) .

Let $K \subset \mathbb{R}^3 \subset S^3 = \mathbb{R}^3 \cup \{\infty\}$ be a knot or a link, where we think of K as a specific embedding, rather than its isotopy class, and assume that with respect to the standard height function in \mathbb{R}^3 there are finitely many local maxima of K occurring on small subarcs μ_1, \dots, μ_r of K . We allow the degenerate case that such a μ_i is a horizontal arc, and we assume that the arcs μ_i are labeled so that $i > j$ implies that the height of μ_i is bigger than or equal to the height of μ_j . We consider a (large) horizontal disk Δ above K and connect every μ_i by a monotonically ascending arc v_i to Δ (see Fig. 12 below). We require that all v_i are pairwise disjoint and do not meet K other than at their lowest point (*the initial point*).

We first want to show that the complement H_2 of the handlebody $H_1 = N(K \cup \{v_1, \dots, v_r\} \cup \Delta) \subset S^3$ is also a handlebody: This can be seen by contracting each v_i while moving μ_i monotonically upward, until it hits Δ . The result is the disk Δ with $2r$ strands attached on its bottom side which descend monotonically until they reach a local minimum of K . These strands

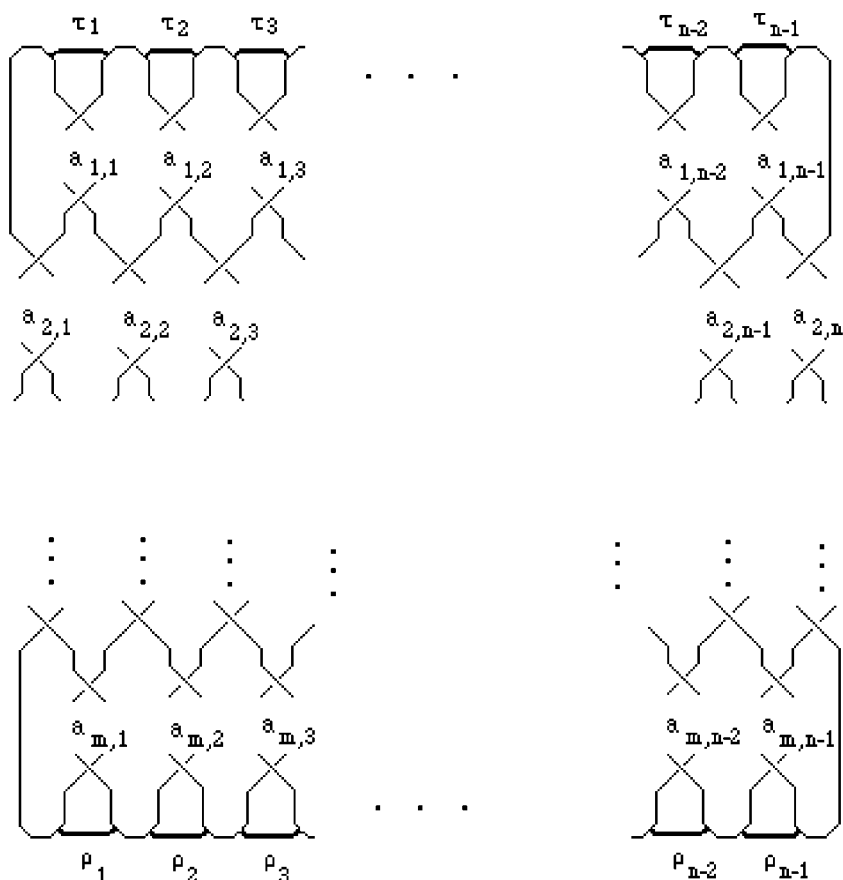


Fig. 11.

are braided, but there is an ambient isotopy of Δ which moves their endpoints around so that the braid becomes trivial. This moves H_1 into a standard position in S^3 , and hence the complement H_2 is also a handlebody.

Next we want to show that the isotopy class of $\Sigma = \partial H_1 = \partial H_2$ in $\mathbb{R}^3 - \mathring{N}(K)$ does not depend on the particular choice of the arcs v_i : For the top arc v_1 this is clear, as there is only one isotopy class of monotonically ascending arcs. For the second topmost arc v_2 there is more than one possible isotopy class, but it is easy to see that the various choices can be obtained from each other by sliding the terminal point of v_2 over $\Delta \cup v_1$. Similarly we slide v_3 over $\Delta \cup v_1 \cup v_2$ to get all possible isotopy classes for v_3 , and so on. The isotopy class of Σ in $\mathbb{R}^3 - K$ is not changed during these moves, which proves our claim. This justifies the following:

Definition 5.3. The above Heegaard splitting $(H_1 - \mathring{N}(K), H_2)$ of $S^3 - \mathring{N}(K)$ is called the *canonical top Heegaard splitting* of the given knot or link K and is denoted by Σ_{top} . Similarly, if we invert the height function, i.e., replacing maxima by minima and making the other corresponding changes, we define the *canonical bottom Heegaard splitting* of the given knot or link K , denoted by Σ_{bot} . Notice

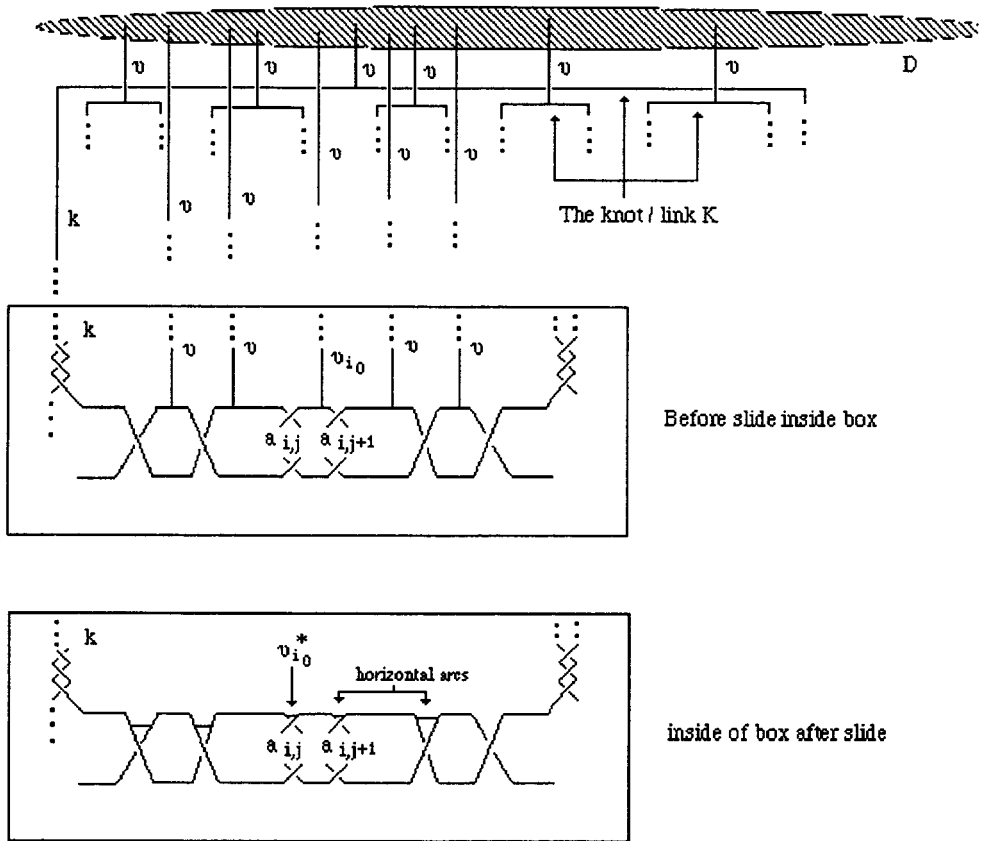


Fig. 12.

that these Heegaard splittings depend on the actual embedding of the curve K in $S^3 = \mathbb{R} \cup \{\infty\}$ and not just on its ambient isotopy class.

We now want to change the viewpoint slightly: Suppose $\{\mu_s, \dots, \mu_t\}$ is a subset of $\{\mu_1, \dots, \mu_r\}$ contained in the same horizontal plane Q . Consider any monotonically ascending subarc k of K which crosses Q transversely and connects it to some μ_d on a strictly higher level. We isotope all of the arcs v_s, \dots, v_t by sliding their terminal point down along v_d and then along k (keeping them throughout pairwise disjoint and their interiors disjoint from K) until they become horizontal arcs v'_s, \dots, v'_t contained in the plane Q . Furthermore we allow iterative slides of any of the v'_i within $Q - K$, over any other v'_j . Again, these slides do not change the isotopy class of the resulting Heegaard splitting.

In this way we obtain an alternative description of the top canonical Heegaard splitting, defining H_1 as neighborhood of K and of a system of horizontal and vertical arcs. In particular this shows the following:

Remark 5.4. For the special case of a $2n$ -plat K the above-defined canonical top Heegaard splitting Σ_{top} coincides, up to an isotopy in $S^3 - \bar{N}(K)$, with the canonical top Heegaard splitting associated with a $2n$ -plat.

We consider now the case of a knot or link K carried by a generalized (!) trellis T and compare its canonical top Heegaard splitting Σ_{top} to the top Heegaard splitting $\Sigma_{\text{top}}(n)$ of the knot projection $K(n)$ obtained from K by an n -flype as defined in Section 3.

Consider the $2n$ local maxima arcs μ_i of $K(n)$ on the same height level which are generated by the n -flype. They are contained in some horizontal plane Q and are connected by vertical arcs v_{i_k} , $k = 1, \dots, 2n$, to the disk Δ . Let μ_{i_0} be the horizontal local maximum arc between the two vertical strands on the interior pair $(e_{i,j}, e_{i,k})$ at which the flype is performed, and let v_{i_0} be the corresponding vertical arc. Now slide these arcs v_{i_k} , $k = 0, \dots, 2n$, as described above so that they become pairwise disjoint horizontal arcs on Q (as indicated in Fig. 12).

Now undo the flype, and obtain a system of arcs $v_{i_k}^*$ with endpoints on $K = K(0)$ which looks as follows: There is a horizontal arc $v_{i_0}^*$ (corresponding to v_{i_0}), together with n vertical arcs on the left and n vertical arcs on the right of the interior pair $(e_{i,j}, e_{i,k})$. The tunnel $v_{i_0}^*$ connects either the two vertical strands winding around $e_{i,j}$, or else those winding around $e_{i,k}$. In the first case (the second is similar) we can slide one endpoint of each of the vertical arcs $v_{i_k}^*$ on the left of $(e_{i,j}, e_{i,k})$ over a subarc of K winding around the edge $e_{i,j}$ and over the arc $v_{i_0}^*$ to transform it into a trivial tunnel (see Fig. 13).

Then we can slide the left endpoint of the arc $v_{i_0}^*$ up along K and some μ_j , then through the disk Δ and over some of the v_i of higher index, and finally back down on some other vertical arc v_h and a subarc of K so that it reaches a position where it is a small horizontal arc which connects the two vertical strands of K which wind around $e_{i,k}$. We then do the same arc slides on the right of $(e_{i,j}, e_{i,k})$ as we did before on the left and hence also transform the other n arcs $v_{i_k}^*$ into trivial tunnels.

This transforms all the $2n$ vertical arcs v_i into trivial tunnels: Each v_i is a small arc with boundary on K which bounds together with a small subarc of K a small disk in H_2 and hence meets a cocore disk in H_1 transverse to v_i precisely in one point. A Heegaard splitting of a 3-manifold M obtained from a given Heegaard splitting by adding a pair of 1-handles, one to each handlebody, so that their cocore disks intersect in a single point is called a *stabilization* of the given Heegaard splitting. We obtain:

Proposition 5.5. *If the knot or link K is carried by a generalized trellis T , and if $K(n)$ is obtained from K by an n -fold flype at some interior pair, then the canonical top Heegaard splitting of $K(n)$ arises from that of K by $2n$ stabilizations. \square*

Box after undoing the tunnels on the left and
before undoing the tunnels on the right.

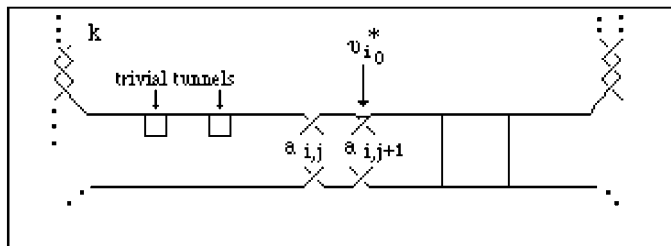


Fig. 13.

We finish this section by considering the change of the canonical top Heegaard splitting induced by adding vertical tunnels:

Lemma 5.6. *Let $K \subset \mathbb{R}^3$ be a knot or link, and let $\{v_i, v'_j\}$ be a set of horizontal or vertical arcs which determine the canonical top Heegaard splitting. Let $\{\alpha_j\}$ be a set of horizontal arcs with endpoints on K which are pairwise disjoint and meet $K \cup \{v_i, v'_j\}$ only in their endpoints. Then the resulting surface $\partial N(K \cup \{v_i\} \cup \{\alpha_j\})$ is a Heegaard surface of $S^3 - N(K)$, and it arises from multiple stabilization of the canonical top Heegaard surface Σ_{top} for K .*

Proof. We first bring the horizontal arcs v'_j into a monotonically ascending position v_j , by successively sliding one of their endpoints over some of the other v'_k , some of the μ_i , and one of the v_k until it reaches Δ . This can be done without changing the position of the α_j . Next we contract the arcs v_i by sliding the μ_i up until they hit Δ (as described in the beginning of the section). We then move the α_i iteratively up, starting always with the topmost one, until they reach Δ . There they form a collection of trivial arcs with endpoints on Δ , which proves the lemma. \square

6. Stabilizing horizontal Heegaard splittings

It is well known that any two Heegaard surfaces Σ and Σ' of M become isotopic after a sufficiently large number of stabilizations on both Heegaard surfaces. If $q \geq 0$ or fewer such stabilizations on either surface suffice for such an isotopy, we will say that Σ is q -isotopic to Σ' . In general it is difficult to determine the minimal possible such q ; an upper bound depending linearly on the genus of the two surfaces has been given recently in [16].

Note: Throughout this section we will always assume that K is a knot.

Part (a) of the following statement seems to be known; for completeness we include a proof.

Lemma 6.1.

- (a) *Let $K \subset M$ be a knot. Every Heegaard surface Σ of the pair (M, K) is 1-isotopic in M to a vertical Heegaard surface of (M, K) and in particular to a Heegaard surface Σ^* of $M - N(K)$.*
- (b) *Let $K \in S^3$ be a knot and Σ a Heegaard surface of the pair (S^3, K) . Let K' be the core of the surgery filling in $\Sigma(\frac{1}{k})$, and Σ_k the Heegaard surface of the pair $(\Sigma(\frac{1}{k}), K')$ defined by Σ as in Section 1. Then Σ^* and Σ_k^* , defined as in part (a) for $M = S^3$ and $M = \Sigma(\frac{1}{k})$ respectively, are isotopic to each other in $S^3 - N(K) = M - N(K')$, for all $k \in \mathbb{Z}$.*

Proof. (a) Let H_1 and H_2 be the two handlebodies of the Heegaard splitting of M given by Σ . We choose a regular neighborhood $N(K) \subset M$ and a meridional disk D for $N(K)$. Consider the arc $\tau = \partial D \cap H_1$. It is properly embedded in H_1 and has endpoints on different sides of K in $\Sigma \cap N(K)$, see Fig. 14 below. We drill out a small neighborhood $N(\tau)$ of τ from H_1 and add it to H_2 , to obtain two new handlebodies H_1^*, H_2^* with common boundary $\partial(H_1 - N(\tau)) = \partial(H_2 \cup N(\tau))$. This defines a new Heegaard splitting of the pair (M, K) which is of genus one higher than the original one. It is

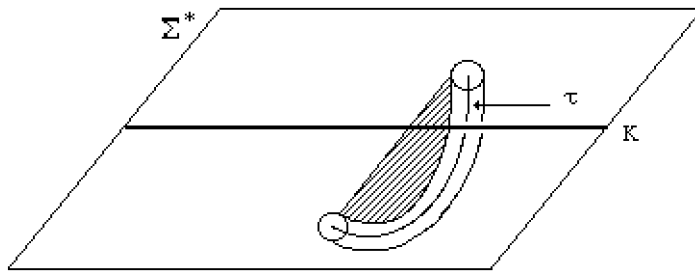


Fig. 14.

a stabilization of Σ , as a cocore disk $D'' \subset N(\tau) \subset H_2^*$ meets the disk $D' = D \cap H_1^*$ exactly once. The disk D' meets K precisely in one point. Hence the new Heegaard surface is a vertical Heegaard surface with respect to K . If we isotope the new surface slightly off K into H_2^* we obtain the desired Heegaard surface Σ^* of $M - \dot{N}(K)$.

(b) We can canonically identify $S^3 - \dot{N}(K)$ with $\Sigma(\frac{1}{k}) - \dot{N}(K')$. After we push the surface Σ off K into $H_2 \subset S^3 - \dot{N}(K)$ it is canonically identified with the surface Σ_k pushed off $N(K')$. This follows from the fact that the closed manifold $\Sigma(\frac{1}{k})$ can be obtained by gluing H_1 to H_2 along their boundaries, but with the gluing map from S^3 modified through a k -fold Dehn twist at K , see Section 1.

As explained above in (a), the surface Σ^* is obtained from Σ using an arc $\tau \subset S^3 - \dot{N}(K) = \Sigma(\frac{1}{k}) - \dot{N}(K')$ with endpoints on Σ , and Σ_k^* is obtained similarly from Σ_k by an analogous arc $\tau_k \subset \Sigma(\frac{1}{k}) - \dot{N}(K') = S^3 - \dot{N}(K)$. These two arcs differ essentially in that τ_k runs once around $\partial N(K)$, as does τ , but in addition τ_k runs k times parallel to K . However, we can define an isotopy between Σ^* and Σ_k^* by sliding one “foot” of $N(\tau)$ k times around a curve K'' on Σ^* which is parallel to K on Σ . \square

Remark 6.2. Notice that Lemma 6.1 remains correct if we replace the knot K by a q -component link L and “1-isotopic” by “ q -isotopic”. This is because one can do the same operations as explained in the last proof, with one stabilization required for each component of L .

Remark 6.3. Let $K \subset S^3$ be a knot carried by some generalized trellis T , and let Σ be the associated trellis Heegaard surface. Then T is a spine of the handlebody $H_1 = N(T)$. After drilling out a properly embedded arc $\tau \subset H_1$ and isotoping the boundary of the new handlebody slightly off K as in the proof of Lemma 6.1(a), we can enlarge the tunnel $N(\tau)$ and thus “peel off” $N(K)$ from H_1 to get a handlebody H'_1 . The boundary $\Sigma' = \partial H'_1$ is isotopic in $S^3 - K$ to the Heegaard surface Σ^* from the proof of Lemma 6.1. Compared to H_1 the new handlebody H'_1 contains an extra handle, namely the neighborhood of the peeled off knot K . The core K of this extra handle is connected to T by a small arc σ which we call the *stem* of the knot, see Fig. 15 below.

For the next proof we need to introduce a new operation on the trellis T , called a *top* (or *bottom*) *horizontal edge slide*. It consists of taking the top (or the bottom) vertex w of a vertical edge e which is the outer-most (say left-most) vertex on some horizontal line of T , and sliding w along that

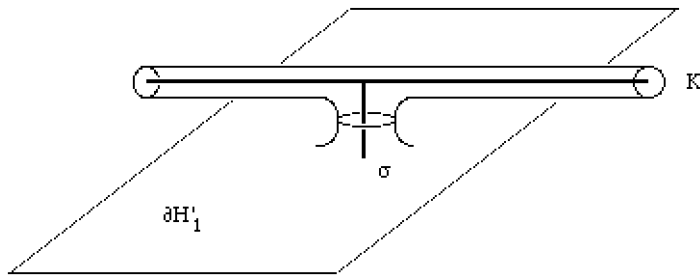


Fig. 15.

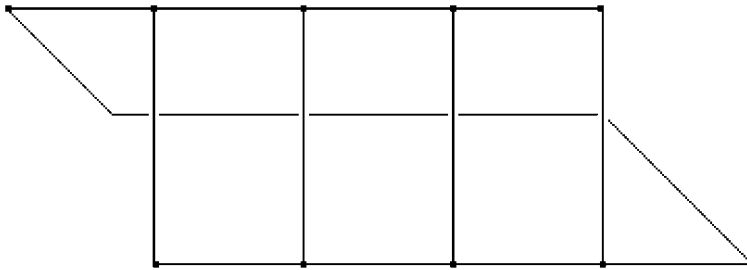


Fig. 16.

horizontal line to the other end. The edge e is isotoped into a position behind the original trellis T , and its top (bottom) endpoint is now the right-most endpoint of the new horizontal line. A picture is given in Fig. 16.

Notice that whenever K is carried by T , such a horizontal edge slide will induce an isotopy of K , as we keep K on $\partial N(T)$ throughout the edge slide. With respect to the new *twisted trellis*, obtained after the horizontal edge slide, the long arc of K on the back of the old trellis (at the height level of the horizontal line of T along which the horizontal edge slide was performed) has now become a short right-most horizontal arc on the front of the twisted trellis, while the former left-most short horizontal arc on the front has now become a long horizontal arc on the back.

Lemma 6.4. *Let K be a knot carried by a generalized trellis T , and let v be the left-most vertex of the top horizontal line of T . Then there exists a finite sequence of horizontal edge slides on T such that the resulting twisted trellis T' has the following property:*

If one starts to slide a point $x \in K$ close to v along the subarc of K which runs parallel to the whole length of the top horizontal line of T' and then once around K , then, for every horizontal layer of T' , x will completely traverse the top and the bottom horizontal line, before it ever crosses over more than one vertical edge (called a “special edge”) from that horizontal layer.

Proof. The point x starts moving along K by traveling first along all of the top horizontal line of T , and then down, winding around the right-most vertical edge e of the first horizontal layer.

According to whether the twist coefficient of e is odd or even, the point x has to continue by sliding towards the left or towards the right. Correspondingly we apply a top horizontal edge slide to all vertical edges of the second layer which have their top endpoints to the left (or to the right) of the bottom vertex of the special edge e , and similarly a bottom horizontal edge slide to all edges of the first layer which have their bottom endpoints to the left (or to the right) side of the bottom vertex of e .

As a consequence x ends up on the subarc of K which completely traverses the second horizontal line of K , and we have to consider the possibility that the endpoint of this horizontal line is the bottom endpoint of a vertical edge in the first layer. In this case x will move again up into this first layer, until it eventually reaches a vertex on the second horizontal line which contains the top endpoint of a vertical edge e' from the second horizontal line. Then x slides down on K into the second layer, winding around the special edge e' , and we have to repeat the procedure just explained, with e' replacing e . This is repeated finitely many times until we have swept out all horizontal lines of K . \square

Proposition 6.5. *For any knot $K \subset S^3$, carried by a generalized trellis T , the associated trellis Heegaard surface Σ is 1-isotopic in $\Sigma\left(\frac{1}{k}\right) = K\left(\frac{1 + ka(K)}{k}\right)$, for any $k \in \mathbb{Z}$, to a multiple stabilization of the canonical top or bottom Heegaard surface Σ_{top} or Σ_{bot} associated to K .*

Proof. We first change the trellis T (and the knot K accordingly) by doing horizontal edge slides so that it satisfies the conclusion of Lemma 6.4. Let Σ be the trellis Heegaard surface for the pair (S^3, K) given by the resulting twisted trellis, still called T , and let K' be the core of the surgery filling of $\Sigma\left(\frac{1}{k}\right)$. We stabilize Σ in $\Sigma\left(\frac{1}{k}\right)$ to get the vertical Heegaard surface Σ^* of $\Sigma\left(\frac{1}{k}\right) - \dot{N}(K') = S^3 - \dot{N}(K)$ as in Lemma 6.1 (a). By Lemma 6.1 (b) this is isotopic in $S^3 - N(K)$ to the Heegaard surface Σ' defined by H'_1 in Remark 6.3. Let σ be the stem as defined there. We will prove the proposition by describing a sequence of slides of the edges of T and of σ . We always think of Σ' as of the boundary of a small regular neighborhood of the handlebody spine which is isotoped along throughout the sequence of slides.

We first introduce a slide of the stem σ in S^3 which keeps the knot endpoint of σ on K and the trellis endpoint on T . This can be done in such a way that the stem is always a short straight arc, for example by keeping it throughout the slide perpendicular to the edge of the trellis along which the trellis endpoint of σ is moving. In particular this shows that we can freely choose the starting position of σ . We choose as starting vertex for the trellis endpoint of σ the top left corner vertex v of T , and for the knot endpoint the point x given by Lemma 6.4.

The *stem slide* is now defined by sliding σ in the described fashion so that its knot endpoint moves around K exactly once. Note that, by the time it comes back to v , every edge e of T has been traversed precisely twice by the trellis endpoint of σ .

Next we introduce, for every edge e of T , a *second coming slide* as follows: Immediately after traversing e for the second time, i.e., with the stem positioned at the “second” endpoint x of e , we interrupt the above stem slide. We isotope the edge e of the trellis along the knot, by sliding its endpoint x first over the stem σ and then back along K , so that e is now replaced by a new stem which is attached to the other endpoint of the former edge e . As this is done after the second and

last time the trellis endpoint visits the edge e , we can complete the above stem slide of σ once around K , although the edge e is now missing in T .

We define the *second coming procedure* as follows: Perform the stem slide, but as the stem σ slides around K do the second coming slide to every edge e of T . This creates lots of new stems and eliminates eventually all edges of the trellis. We now investigate more precisely the effect of this second coming procedure on the edges of the trellis:

Before this procedure, vertical edges of the trellis had either 2, 3 or 4 adjacent horizontal edges, depending on their location in the trellis. Consider a vertical edge e which had 4 adjacent horizontal edges. Notice that it follows from the horizontal edge slides performed at the beginning of the proof in accordance with Lemma 6.4 that in the stem slide the trellis endpoint of σ traverses each of the 4 horizontal edges at least once before it crosses over the vertical edge e for the first time. Hence the first passage of the trellis endpoint through e will produce precisely one stem at one of the endpoints of e , and none at the other. The second passage through e will produce a stem at each of the two endpoints of e . Thus for every vertical edge e with 4 adjacent horizontal edges the second coming procedure gives precisely a single stem at one of the endpoints of e and a double stem at the other. Note that this double stem has none of its endpoints on the string of the knot K which runs horizontally on the back of Σ . Instead, it connects the two strings of K which wind around the vertical edge e .

By similar considerations the same conclusion holds for vertical edges with 3 adjacent horizontal edges, if the analogous assumption is satisfied. This includes the horizontal edge in the first layer with v as top vertex (even if it has only 2 adjacent horizontal edges), as can be seen directly from the fact that the original stem, in final position, will be placed with trellis endpoint at v .

Observe now that, as a consequence of the horizontal edge slides performed on T and K at the beginning of this proof, the only vertical edges in T with only 2 adjacent horizontal edges are possibly the special edges from Lemma 6.4 or the edge with endpoint v . Hence, by the time the trellis endpoint of σ has returned to the starting vertex v , there will be a double stem at the top or the bottom of the corresponding twist box for all vertical edges except for the one special edge in every layer.

The Heegaard surface Σ'' isotopic to Σ' which results from the second coming procedure is hence obtained from K by introducing the tunnel system given by all the double stems (the single ones can be deleted without changing the isotopy class of Σ''). Thus it follows from Lemma 5.6 that Σ'' is isotopic to a multiple stabilization of either the top or the bottom vertical Heegaard surface Σ_{top} or Σ_{bot} . \square

We can now apply the above proposition to knots with flypes and obtain:

Theorem 6.6. *Let T be a generalized trellis with an interior pair of edges, and let $T(n)$ be the trellis obtained from an n -flype at this edge pair. Let $K \subset S^3$ be a knot carried by T . Then for all $n \in \mathbb{Z}$ the trellis Heegaard surface $\Sigma(n) \subset K((1 + ka(K))/k)$ of genus $g(T(n)) = g(T) + 2n$, given by the trellis $T(n)$, is 1-isotopic to a multiple stabilization of the canonical top or bottom Heegaard surface Σ_{top} or Σ_{bot} of $K((1 + ka(K))/k)$, defined by the trellis projection of K before the flypes.*

Proof. This is an immediate consequence of the last Proposition 6.5. and of Proposition 5.5. \square

Remark 6.7. An alternative proof of the last theorem can be given by combining a result of Sedgwick [19] with Proposition 6.5. Notice that Sedgwick's proof applies to a more general situation than the one given by trellisses, since it is a local proof. On the other hand, it is not possible to deduce the statement of Proposition 6.5 by his methods, as that statement is of global nature.

Proof of Theorem 0.3. The theorem follows directly from Proposition 6.5 and Theorem 6.6. \square

7. Stabilizing canonical vertical Heegaard splittings

In this section we investigate the question of how many stabilizations are necessary so that the canonical top and bottom Heegaard splittings of a knot $K \subset S^3$, given as a $2n$ -plat, become isotopic. It was proved by Hagiwara [5] that $n - 1$ stabilizations always suffice. We give a new proof of this result and show that in many cases one can do with considerably fewer stabilizations. The following notion has been introduced, with minor technical variations, in [11].

Definition 7.1. (a) A $2n$ -braid b will be said to have *total width* $r \in \mathbb{N}$ if in its standard projection $\pi(b) \subset P$ (obtained from b by replacing every crossing by a node), every monotonically descending path in $\pi(b)$ connecting the i th strand at the top to the k th strand on the bottom satisfies $i - k \leq 2r + 1$, and r is the smallest such number.

(b) A $2n$ -plat projection of a knot K will be said to have *total width* $r \in \mathbb{N}$ if the underlying $2n$ -braid has total width r .

Clearly for any $2n$ -plat one always has $0 \leq r \leq n - 1$. If $r = 0$ then the braid in question defines the n -component unlink. If m is the number of horizontal layers of the plat, then one has $r \leq (m + 1)/2$.

We prove

Proposition 7.2. *For every knot or link $K \subset S^3$ of width r the two canonical Heegaard splittings of $S^3 - \dot{N}(K)$ are r -isotopic.*

Proof. For each i with $r + 1 \leq i \leq n - 1$ we consider the tunnel system consisting of the tunnels $\tau_1, \dots, \tau_{i-1}, \rho_{i-r}, \rho_{i-r+1}, \dots, \rho_{n-1}$. We claim that this system is isotopic to the system $\tau_1, \dots, \tau_{i-2}, \rho_{i-r}, \rho_{i-r+1}, \dots, \rho_{n-1}, \eta$, where η is a trivial tunnel. Assuming this claim it follows from the symmetry between the top and bottom tunnels that the system

$$\tau_1, \dots, \tau_{i-2}, \rho_{i-r}, \rho_{i-r+1}, \dots, \rho_{n-1}, \eta$$

is isotopic to the system

$$\tau_1, \dots, \tau_{i-2}, \rho_{i-r-1}, \rho_{i-r}, \dots, \rho_{n-1}.$$

Thus we conclude inductively that

$$\tau_1, \dots, \tau_{n-1}, \rho_{n-r}, \rho_{n-r+1}, \dots, \rho_{n-1}$$

is isotopic to

$$\tau_1, \dots, \tau_r, \rho_1, \rho_2, \dots, \rho_{n-1}.$$

It follows from Lemma 5.6 that these systems are just the canonical systems with r trivial tunnels added.

It remains to prove the above claim. The assumption on the total width of the $2n$ -plat K implies that there is no monotonically descending path connecting the $2i$ th strand on the top to the $(2(i-r)-2)$ th strand on the bottom of the plat. In other words, the leftmost monotonically descending path γ in $\pi(K)$ which starts at the top of the $2i$ th string must end at the bottom of some k th string with $k \geq 2(i-r)-1$. We consider the handlebody $W = N(K \cup \tau_1 \cup \dots \cup \tau_{i-1} \cup \rho_{i-r} \cup \rho_{i-r+1} \cup \dots \cup \rho_{n-1})$ and, for all $j = i-r, \dots, n-1$, we introduce cocore disks D_j for the tunnels ρ_j (see Fig. 17).

Consider now an equatorial 2-sphere S which intersects K just below the top bridges and cuts off a 3-ball B with n unknotted arcs t_1, \dots, t_n (the top bridges), as indicated in Fig. 17.

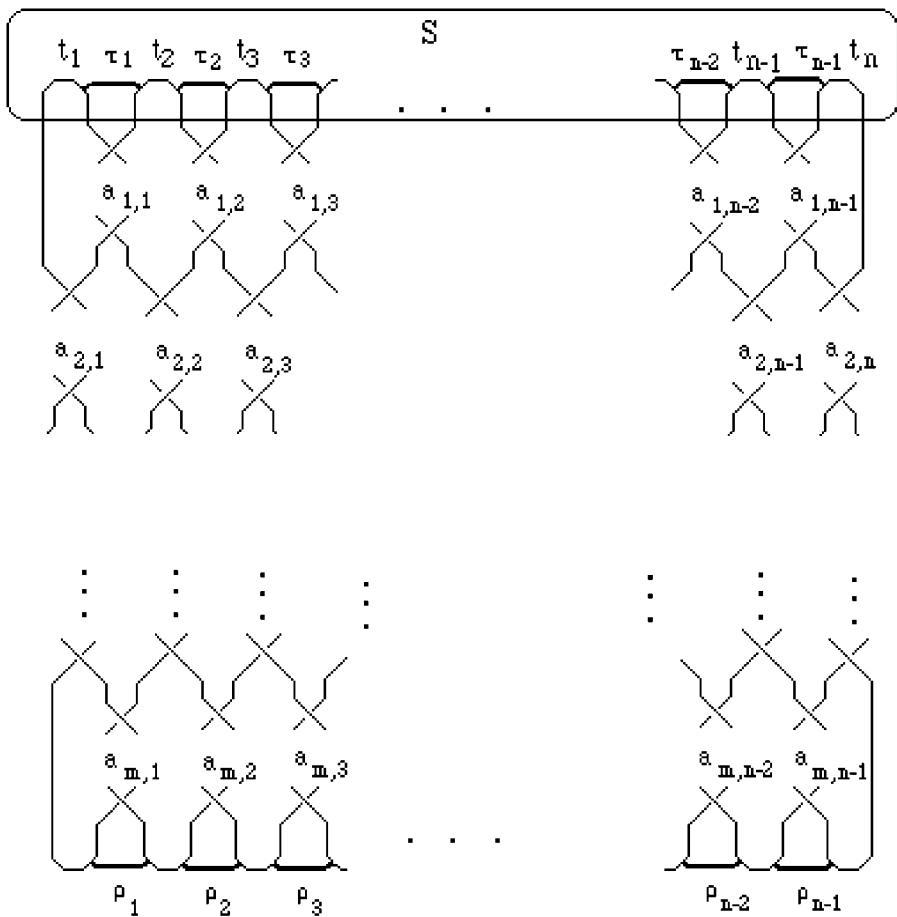


Fig. 17.

Let β be an arc in S which is isotopic relative boundary to the top tunnel τ_i . Consider an isotopy \mathcal{I} of S determined by moving the sphere monotonically down, so that at each level we have a horizontal 2-sphere, to a level just above the the bottom bridges. The isotopy \mathcal{I} moves the intersection points $\bigcup_{j=1}^n \{t_j \cap S\}$ in such a way on S that it braids the arcs t_j according to the strands of the given $2n$ -plat K . Let β' be the image of β after the isotopy \mathcal{I} .

We can assume without loss of generality that each crossing of $\pi(K)$ lies on a distinct height level, called a “critical” level. The leftmost descending path γ in $\pi(b)$, defined above, determines at each horizontal level a split of S into “left” and “right” halves. (To be precise, $S - \{\infty\}$ is split along $\pi^{-1}(\gamma)$, where $\pi: \mathbb{R}^3 \rightarrow P$ is the orthogonal projection.) As we move S by the isotopy \mathcal{I} through a critical level we see it relatively that \mathcal{I} can be chosen so that the arc β is always contained in an ε -neighborhood of the right half of S determined by γ , where ε is smaller than the distance between any two strands of the plat. In particular, when S has reached the bottom level, then the obtained arc β' is positioned entirely to the right of the $(2(i-r)-2)$ th strand. Thus we can isotope β' on ∂W to become a small trivial arc η by sliding it across the cocore disks D_{i-r}, \dots, D_{n-1} of the tunnels $\rho_{i-r}, \dots, \rho_{n-1}$. This proves the claim and finishes the proof of the proposition. \square

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